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Comma vertex and string field algebra

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Abstract: We study the matter part of the algebra of open string fields using the 3-string vertex over the sliver state, which we call "comma vertex". By generalizing this comma vertex to the N-string overlap, we obtain a closed form of the Neumann coefficients in the N-string vertex and discuss its relation to the oscillator representation of wedge states.

Keywords: Bosonic Strings, D-branes.

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Contents

1. Introduction

Recently, the cubic open string field theory [1] around the tachyon vacuum, or the "vacuum string field theory" [2, 3] was proposed. This theory contains no physical open string excitation and is conjectured to describe the appearance of closed strings after the open string tachyon condensation. The matter part of the solution of the classical equation of motion around this vacuum is given by projection elements in the star algebra of open string fields [4].

As an example of projection, there is a state called sliver [6], which is special in a sense that this state can be defined in an arbitrary boundary CFT. In the case of flat background a projection was also constructed in the oscillator formalism [5] and was identified with the sliver with numerical evidence [4]. To construct projections more systematically, it is important to find a matrix representation of open string field [7, 8, 9] (see [10]–[13] for related issues). The essential ingredient behind this matrix representation is the so-called "comma" form of the overlap vertex [14, 15, 16]. Interestingly, if we perform a Bogoliubov transformation to make the sliver state to be a new Fock vacuum, the 3-string vertex takes a comma form [5]. In this respect the sliver is a key to the structure of the string field algebra. In this paper, we study the properties of the overlap vertices constructed over the sliver state, and discuss some applications.

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This paper is organized as follows: in section 2, we consider the 3-string vertex over the sliver state. In section 3, we analyze the structure of string field algebra using this vertex. In section 4, we construct the oscillator representation of wedge states and find a closed form of the Neumann coefficient of N-string vertex. Section 5 is devoted to the discussions.

2. Comma vertex

In this section, we review the 3-string vertex constructed over the sliver state [5]. In the following, we will call this vertex the "comma vertex" [14, 15, 16].

2.1 3-string Neumann coefficient

The matter part of the 3-string vertex in zero-momentum sector is given by [17, 18, 19]

$$|V_3\rangle = \mathcal{N}_{V_3}^{26} \exp\left(-\frac{1}{2}\sum_{\substack{r,s=1,2,3\\ m,n\geq 1}} a_m^{r\dagger} V_{3\,mn}^{rs} a_n^{s\dagger}\right)|0\rangle_{123}\,. \tag{2.1}$$

$\mathcal{N}_{V_3}$ is a normalization constant. In this paper we concentrate on zero-momentum sector but algebraic relations we will obtain are the same if we consider whole sector [5, 4, 8]. The identity string field $|I\rangle$ and the reflector $\langle R(1,2)|$ is given by (up to normalization)

$$|I\rangle = e^{-\frac{1}{2}a^\dagger C a^\dagger}|0\rangle\,, \qquad \langle R(1,2)| = {}_{12}\langle 0|e^{-a_1 C a_2}\,, \tag{2.2}$$

where $C_{nm} = (-1)^n \delta_{nm}$. We define star product of string fields as

$$|A \star B\rangle_1 = {}_3\langle r(A)|_2\langle r(B)||V_3\rangle_{123}\,, \tag{2.3}$$

where

$${}_1\langle r(A)| = \langle R(1,2)|A\rangle_2\,. \tag{2.4}$$

As preliminaries, we list some useful formulas satisfied by the Neumann coefficients V_3^{rs} in the 3-string vertex. We introduce the matrices X, Y, Z by

$$X = CV_3^{11}, \quad Y = CV_3^{12}, \quad Z = CV_3^{21}\,. \tag{2.5}$$

It can be shown that these matrices commute with each other and satisfy the following relations:

$$X + Y + Z = 1\,, \tag{2.6}$$

$$X^2 + Y^2 + Z^2 = 1\,, \qquad YZ = X^2 - X\,, \tag{2.7}$$

$$Y^3 + Z^3 = 2X^3 - 3X^2 + 1\,, \tag{2.8}$$

$$CXC = X\,, \qquad CYC = Z\,. \tag{2.9}$$

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Using these matrices, one can construct the orthogonal projections L and R[1]

$$L = \frac{Y(1-TX)+TZ^2}{(1+T)(1-X)}\,, \qquad R = \frac{Z(1-TX)+TY^2}{(1+T)(1-X)}\,, \tag{2.10}$$

which satisfy

$$L+R=1\,, \qquad L^2=L\,, \qquad R^2=R\,, \qquad LR=0\,, \qquad L=CRC\,. \tag{2.11}$$

Here T is defined by

$$T = \frac{1}{2X}\left(1+X-\sqrt{(1-X)(1+3X)}\right)\,, \tag{2.12}$$

and satisfies

$$XT^2-(1+X)T+X=0\,. \tag{2.13}$$

This matrix T appears in the sliver state [5, 4]

$$|\Xi\rangle = \mathcal{N}_\infty^{26}\exp\left(-\frac{1}{2}a^\dagger CTa^\dagger\right)|0\rangle\,, \tag{2.14}$$

which satisfies $|\Xi\rangle\star|\Xi\rangle = |\Xi\rangle$. The matrices Y and Z are decomposed into L and R as

$$Y = L(1-TX)+RX(T-1) = (1-TX)(L-TR)\,, \tag{2.15}$$
$$Z = LX(T-1)+R(1-TX) = (1-TX)(R-TL)\,. \tag{2.16}$$

2.2 Vertex over the sliver

We introduce new oscillators s which annihilate the sliver by performing a Bogoliubov transformation from the original oscillators a

$$s = \frac{1}{\sqrt{1-T^2}}(a+CTa^\dagger)\,, \qquad s^\dagger = \frac{1}{\sqrt{1-T^2}}(a^\dagger+CTa)\,, \tag{2.17}$$
$$s|\Xi\rangle = 0\,. \tag{2.18}$$

The 3-string vertex in this new basis (up to normalization) turns out to be [5]

$$|V_3\rangle = \exp\left(-\sum_{r=1}^{3}s_r^\dagger CLs_{r+1}^\dagger\right)|\Xi\rangle_{123} = \exp\left(-\frac{1}{2}\sum_{r,s=1}^{3}s_r^\dagger\widehat{V}_3^{rs}s_s^\dagger\right)|\Xi\rangle_{123}\,, \tag{2.19}$$

where $\widehat{V}_3^{rs}$ is given by

$$C\widehat{V}_3 = (1-CV_3T)^{-1}(CV_3-T) = \begin{pmatrix} 0 & L & R \\ R & 0 & L \\ L & R & 0 \end{pmatrix}\,. \tag{2.20}$$

[1]In [7], L and R were denoted ρ_1 and ρ_2, respectively.

The identity string field and the reflector in this new basis are found to be

$$|I\rangle = e^{-\frac{1}{2}s^\dagger C s^\dagger}|\Xi\rangle\,, \qquad \langle R(1,2)| = {}_{12}\langle\Xi|e^{-s_1 C s_2}\,. \tag{2.21}$$

Using the projections L and R, we can define the splitting of s into two parts

$$s = \sum_\alpha s_{L\alpha} e_\alpha + s_{R\alpha} f_\alpha\,, \tag{2.22}$$

where e_α and f_α are the orthonormal basis of the eigenspace of L and R:

$$\begin{aligned} &e_\alpha \cdot e_\beta = f_\alpha \cdot f_\beta = \delta_{\alpha\beta}\,, && e_\alpha \cdot f_\beta = 0\,, && f_\alpha = -Ce_\alpha\,, \\ &Le_\alpha = e_\alpha\,, && Re_\alpha = 0\,, && Lf_\alpha = 0\,, \qquad Rf_\alpha = f_\alpha\,, \\ &L = \textstyle\sum_\alpha e_\alpha e_\alpha^T\,, && R = \textstyle\sum_\alpha f_\alpha f_\alpha^T\,. \end{aligned} \tag{2.23}$$

Since s_L and s_R commute,

$$[s_{L\alpha}, s_{L\beta}^\dagger] = [s_{R\alpha}, s_{R\beta}^\dagger] = \delta_{\alpha\beta}\,, \qquad [s_{L\alpha}, s_{R\beta}^\dagger] = 0\,, \tag{2.24}$$

the Hilbert space factorizes into the Fock spaces of s_L and s_R

$$\mathcal{A}_{\rm str} = \mathcal{H}_L \otimes \mathcal{H}_R\,. \tag{2.25}$$

In terms of $s_{L,R}$, the 3-string vertex takes a simple form ("comma" form)

$$|V_3\rangle = \exp\left(s_{1R}^\dagger s_{2L}^\dagger + s_{2R}^\dagger s_{3L}^\dagger + s_{3R}^\dagger s_{1L}^\dagger\right)|\Xi\rangle_{123}\,, \tag{2.26}$$

where

$$s_L^\dagger s_R^\dagger = \sum_\alpha s_{L\alpha}^\dagger s_{R\alpha}^\dagger\,. \tag{2.27}$$

The identity string field and the reflector become

$$|I\rangle = e^{s_R^\dagger s_L^\dagger}|\Xi\rangle\,, \qquad \langle R(1,2)| = {}_{12}\langle\Xi|e^{s_{1R}s_{2L}+s_{2R}s_{1L}}\,. \tag{2.28}$$

2.3 On the structure of the vertex over the sliver

The vanishing of the diagonal components of $\widehat{V}_3$ in (2.20) is simply explained from the identity $|\Xi\rangle \star |\Xi\rangle = |\Xi\rangle$ written in terms of the s-oscillators [5]. The identities $LR = 0$ and $RL = 0$, which seem somewhat accidental from the complicated expression (2.10), also simply follow from $|\Xi\rangle \star |I\rangle = |\Xi\rangle$ and $|I\rangle \star |\Xi\rangle = |\Xi\rangle$ in the s-oscillator basis:

$$\begin{aligned} \left(|\Xi\rangle \star |I\rangle\right)_1 &= {}_3\langle\Xi|_2\langle I||V_3\rangle_{123} \\ &= {}_3\langle\Xi|_2\langle\Xi|\exp\left(-\frac{1}{2}s_2 C s_2\right)\exp\left(-\frac{1}{2}\sum_{r,s=1}^{3} s_r^\dagger \widehat{V}_3^{rs} s_s^\dagger\right)|\Xi\rangle_{123} \\ &= {}_2\langle\Xi|\exp\left(-\frac{1}{2}s_2 C s_2\right)\exp\left(-s_1^\dagger \widehat{V}^{12} s_2^\dagger\right)|\Xi\rangle_{12} \\ &= \exp\left(-\frac{1}{2}s_1^\dagger \widehat{V}^{12} C \widehat{V}^{21} s_1^\dagger\right)|\Xi\rangle_1\,, \end{aligned} \tag{2.29}$$

where use has been made for the formula

$$\langle 0|\exp\left(\lambda_i a_i - \frac{1}{2}P_{ij}a_i a_j\right)\exp\left(\mu_i a_i^\dagger - \frac{1}{2}Q_{ij}a_i^\dagger a_j^\dagger\right)|0\rangle = \tag{2.30}$$
$$= \det K^{-1/2}\exp\left(\mu K^{-1}\lambda - \frac{1}{2}\lambda Q K^{-1}\lambda - \frac{1}{2}\mu K^{-1}P\mu\right), \qquad K \equiv 1 - PQ\,.$$

From eq. (2.29) we observe $|\Xi\rangle \star |I\rangle = |\Xi\rangle$ implies $\widehat{V}^{12}C\widehat{V}^{21} = 0$, which is equivalent to $LR = 0$. Similarly, $RL = 0$ can be understood from $|I\rangle \star |\Xi\rangle = |\Xi\rangle$.

We briefly comment on other identities in (2.11). $\widehat{V}_3$ can be diagonalized in the r, s indices by a matrix $\mathcal{O}$ satisfying $\mathcal{O}^{-1} = \mathcal{O}^\dagger$:

$$\widehat{V}_3 = \mathcal{O}^{-1}\widehat{V}_D\mathcal{O}\,, \tag{2.31}$$

with

$$\widehat{V}_D \equiv \begin{pmatrix} C & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \bar{\Lambda} \end{pmatrix}, \qquad \mathcal{O} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ \omega & \bar{\omega} & 1 \\ \bar{\omega} & \omega & 1 \end{pmatrix}. \tag{2.32}$$

Here $\omega \equiv \exp(2\pi i/3)$. $\widehat{V}_D^{11} = C$ follows because this component represents identity string field type overlap condition (see (2.2) and (2.21)). From (2.31) and (2.32), we obtain

$$\begin{aligned} C &= \widehat{V}_3^{12} + \widehat{V}_3^{21}\,, \\ \Lambda &= \bar{\omega}\widehat{V}_3^{12} + \omega\widehat{V}_3^{21}\,, \\ \bar{\Lambda} &= \omega\widehat{V}_3^{12} + \bar{\omega}\widehat{V}_3^{21}\,, \end{aligned} \tag{2.33}$$

and $C = \widehat{V}_3^{12} + \widehat{V}_3^{21}$ is equivalent to $L + R = 1$. Notice that $L^2 = L$ and $R^2 = R$ follow from $L + R = 1$ and $LR = RL = 0$.

Finally the equation $L = CRC$ follows from the property of the original vertex $CV^{12} = V^{21}C$, since C and T which appear in the Bogoliubov transformation (2.17) commute with C. $CV^{12} = V^{21}C$ describes a property of the vertex under the string worldsheet orientation reversal.

3. String field algebra

In this section, we discuss the structure of the string field algebra $\mathcal{A}_{\mathrm{str}}$. The results in this section were essentially obtained in [9] using a different representation of the vertex.

Strictly speaking, $\mathcal{A}_{\mathrm{str}}$ in this section should be understood as the zero-momentum subalgebra of string field algebra, since the sliver state does not carry momentum. However, the most of the following discussions can be applied to the whole algebra by replacing the sliver state by the D-instanton state, since the Neumann coefficients including the zero-modes satisfy the same relations as those without zero-modes [5, 4, 8].

3.1 String field oscillators

First, we construct the string field oscillators $A_\alpha, A_\alpha^\dagger$ [9] which satisfy the canonical commutation relations

$$[A_\alpha, A_\beta^\dagger]_\star = \delta_{\alpha\beta}|I\rangle\,. \tag{3.1}$$

To find the explicit expression of the string field oscillators, it is convenient to introduce the coherent states

$$|z_L, z_R\rangle = \exp(z_L s_L^\dagger + z_R s_R^\dagger)|\Xi\rangle\,, \tag{3.2}$$
$$I(z_L, z_R) = \exp(z_L s_L^\dagger + z_R s_R^\dagger)|I\rangle\,. \tag{3.3}$$

The multiplication rules of these coherent states are found to be

$$|z_L, z_R\rangle \star |w_L, w_R\rangle = e^{z_R w_L}|z_L, w_R\rangle\,, \tag{3.4}$$
$$I(z_L, z_R) \star |w_L, w_R\rangle = e^{z_R w_L}|z_L + w_L, w_R\rangle\,, \tag{3.5}$$
$$|z_L, z_R\rangle \star I(w_L, w_R) = e^{z_R w_L}|z_L, z_R + w_R\rangle\,, \tag{3.6}$$
$$I(z_L, z_R) \star I(w_L, w_R) = e^{z_R w_L} I(z_L + w_L, z_R + w_R)\,. \tag{3.7}$$

Using (3.7), we can check that the string fields

$$A_\alpha = s_{R\alpha}^\dagger|I\rangle\,, \qquad A_\alpha^\dagger = s_{L\alpha}^\dagger|I\rangle\,, \tag{3.8}$$

satisfy the relation (3.1). Since the sliver state is annihilated by the string field oscillators

$$A_\alpha \star |\Xi\rangle = |\Xi\rangle \star A_\alpha^\dagger = 0\,, \tag{3.9}$$

the sliver can be identified with a projection onto the Fock vacuum $|0\rangle\rangle$ of the string field oscillators:

$$|\Xi\rangle = |0\rangle\rangle\langle\langle 0|\,. \tag{3.10}$$

This identification is justified by the fact that the sliver is naturally factorized into the vacuum of s_L and s_R:

$$|\Xi\rangle = |\Xi_L\rangle \otimes |\Xi_R\rangle\,, \tag{3.11}$$

where

$$s_{L\alpha}|\Xi_L\rangle = 0\,, \qquad s_{R\alpha}|\Xi_R\rangle = 0\,. \tag{3.12}$$

3.2 Matrix representation of string fields

In this subsection we construct a mapping between open string fields and matrices. Since the identity string field has the form $|I\rangle = e^{s_L^\dagger s_R^\dagger}|\Xi\rangle$, the string field oscillators (3.8) can be rewritten as

$$A_\alpha = s_{L\alpha}|I\rangle = s_{R\alpha}^\dagger|I\rangle\,, \tag{3.13}$$
$$A_\alpha^\dagger = s_{L\alpha}^\dagger|I\rangle = s_{R\alpha}|I\rangle\,. \tag{3.14}$$

Using (3.7) recursively, we can show the following relations:

$$A_{\alpha_1} \star \cdots \star A_{\alpha_k} = s_{L\alpha_1} \cdots s_{L\alpha_k} |I\rangle = s^\dagger_{R\alpha_1} \cdots s^\dagger_{R\alpha_k} |I\rangle \,, \tag{3.15}$$

$$A^\dagger_{\alpha_1} \star \cdots \star A^\dagger_{\alpha_k} = s^\dagger_{L\alpha_1} \cdots s^\dagger_{L\alpha_k} |I\rangle = s_{R\alpha_1} \cdots s_{R\alpha_k} |I\rangle \,, \tag{3.16}$$

$$[A_\alpha, A^\dagger_\beta]_\star = [s_{L\alpha}, s^\dagger_{L\beta}] |I\rangle = [s_{R\alpha}, s^\dagger_{R\beta}] |I\rangle \,. \tag{3.17}$$

Therefore, the identity string field defines a morphism between worldsheet oscillators $s_{L,R}$ and the string field oscillators. This induces the following isomorphism between the Fock spaces:

$$\mathcal{H}_L \cong \mathcal{H}_{\rm str} \,, \qquad \mathcal{H}_R \cong \mathcal{H}^*_{\rm str} \,, \tag{3.18}$$

where $\mathcal{H}_{\rm str}$ is the Fock space of string field oscillators. Finally we have the identification between open string fields and the matrices acting on $\mathcal{H}_{\rm str}$:

$$\mathcal{A}_{\rm str} = \mathcal{H}_L \otimes \mathcal{H}_R = \mathcal{H}_{\rm str} \otimes \mathcal{H}^*_{\rm str} = {\rm End}(\mathcal{H}_{\rm str}) \,. \tag{3.19}$$

We can construct this mapping in a more explicit way. Using the morphism (3.16), the star exponential of string field oscillators turns out to be

$$e_\star^{z_L A^\dagger} = e^{z_L s^\dagger_L} |I\rangle \,, \qquad e_\star^{z_R A} = e^{z_R s^\dagger_R} |I\rangle \,. \tag{3.20}$$

From (3.5), (3.6) and (3.20), we can see that the coherent states of $s_{L,R}$ and A_α are related as

$$|z_L, z_R\rangle = e^{z_L s^\dagger_L} |I\rangle \star |\Xi\rangle \star e^{z_R s^\dagger_R} |I\rangle = e^{z_L A^\dagger} |0\rangle\rangle\langle\langle 0| e^{z_R A} = |z_L\rangle\rangle\langle\langle z_R| \,. \tag{3.21}$$

Note that this relation is consistent with the trace

$${\rm Tr}\left(|z_L, z_R\rangle \right) = \langle I | z_L, z_R\rangle = e^{z_L z_R} \,. \tag{3.22}$$

By expanding (3.21) in terms of $z_{L,R}$, we find that the occupation number state of $s_{L,R}$ corresponds to the matrix element between the number states of string field oscillators:

$$|n, m\rangle = \prod_{\alpha,\beta} \frac{s^{\dagger n_\alpha}_{L\alpha} s^{\dagger m_\beta}_{R\beta}}{\sqrt{n_\alpha! m_\beta!}} |\Xi\rangle = \prod_\alpha \frac{A^{\dagger n_\alpha}_\alpha}{\sqrt{n_\alpha!}} |0\rangle\rangle\langle\langle 0| \prod_\beta \frac{A^{m_\beta}_\alpha}{\sqrt{m_\beta!}} = |n\rangle\rangle\langle\langle m| \,. \tag{3.23}$$

We can see that the number states behave exactly as matrix elements:

$$|n\rangle\rangle\langle\langle m| \star |k\rangle\rangle\langle\langle l| = \delta_{m,k} |n\rangle\rangle\langle\langle l| \,, \tag{3.24}$$

$${\rm Tr}\left(|n\rangle\rangle\langle\langle m| \right) = \delta_{n,m}. \tag{3.25}$$

Now we can construct a mapping between string fields and matrices as follows. Since every state can be expanded in terms of the number basis of $s_{L,R}$, every string field can be written as a matrix:

$$|\Psi\rangle = \sum_{n,m} \Psi_{n,m} |n, m\rangle = \sum_{n,m} \Psi_{n,m} |n\rangle\rangle\langle\langle m| \,. \tag{3.26}$$

Especially, the identity string field corresponds to the identity matrix

$$|I\rangle = e^{s_L^\dagger s_R^\dagger}|\Xi\rangle = \sum_n |n,n\rangle = \sum_n |n\rangle\rangle\langle\langle n|\,. \tag{3.27}$$

Note that the 3-string vertex can also be written in terms of the number basis

$$|V_3\rangle_{123} = \sum_{l,m,n} |l\rangle\rangle\langle\langle m|_1 \otimes |m\rangle\rangle\langle\langle n|_2 \otimes |n\rangle\rangle\langle\langle l|_3\,. \tag{3.28}$$

We can see that this vertex represents the matrix multiplication.

4. Wedge states and N-string vertices

In this section, we first construct oscillator representations of wedge states, which were originally defined as a class of surface states in [6]. Then we present a closed form of the Neumann coefficients in the N-string vertex and discuss their relation to the wedge states.

4.1 Oscillator representation of wedge states

We define the n^{th} wedge state $|n\rangle_w$ by [6]

$$|n\rangle_w \equiv \Big(|0\rangle\Big)_\star^{n-1}\,. \tag{4.1}$$

In this subsection, we will find an oscillator representation of the wedge state under the following ansatz[2]

$$|n\rangle_w = \mathcal{N}_n^{26} \exp\left(-\frac{1}{2}a^\dagger C T_n a^\dagger\right)|0\rangle. \tag{4.2}$$

The recursion relations for $\mathcal{N}_n$ and T_n follow from the relation $|n+1\rangle_w = |n\rangle_w \star |0\rangle$:

$$T_{n+1} = \frac{X(1-T_n)}{1-XT_n}, \tag{4.3}$$

$$\mathcal{N}_{n+1} = \mathcal{N}_n \mathcal{N}_{V_3} \det(1-XT_n)^{-1/2}\,. \tag{4.4}$$

In the above we have assumed that T_n commutes with X, Y and Z which will appear to hold consistently when the explicit form of T_n is obtained. In order for the wedge state to converge in the limit $n\to\infty$, we should set the normalization $\mathcal{N}_{V_3}$ of the 3-string vertex to be

$$\mathcal{N}_{V_3} = \det(1-XT)^{1/2}. \tag{4.5}$$

[2]It was informed to us by B. Zwiebach that the oscillator representation of wedge states has been obtained independently by A. Sen.

By rewriting the recursion relations using (2.13)

$$\frac{1}{T_{n+1}-T}-\frac{T}{1-T^2}=-\frac{1}{T}\left(\frac{1}{T_n-T}-\frac{T}{1-T^2}\right), \tag{4.6}$$

$$\frac{\mathcal{N}_{n+1}}{\mathcal{N}_n}=\det\left(-\frac{1}{T}\frac{T_{n+1}-T}{T_n-T}\right)^{1/2}, \tag{4.7}$$

and putting the initial value $T_2=0$, $\mathcal{N}_2=1$, we find

$$T_n=\frac{T+(-T)^{n-1}}{1-(-T)^n}, \tag{4.8}$$

$$\mathcal{N}_n=\det\left[\frac{1-T^2}{1-(-T)^n}\right]^{1/2}. \tag{4.9}$$

Note that the normalization of the identity $\mathcal{N}_1$ and the 3-string vertex $\mathcal{N}_{V_3}$ satisfy the relation found in [8]:

$$\mathcal{N}_1\mathcal{N}_{V_3}=\det(1-X)^{1/2}. \tag{4.10}$$

Note also that $\mathcal{N}_{V_3}$ is equal to $\mathcal{N}_3$. The final expression of the wedge state is

$$|n\rangle_w=\det\left[\frac{1-T^2}{1-(-T)^n}\right]^{26/2}\exp\left(-\frac{1}{2}a^\dagger C\frac{T+(-T)^{n-1}}{1-(-T)^n}a^\dagger\right)|0\rangle\,. \tag{4.11}$$

In this normalization, the sliver state which can be defined as a limit of the wedge states [6] becomes

$$|\Xi\rangle=\lim_{n\to\infty}|n\rangle_w=\det(1-T^2)^{26/2}\exp\left(-\frac{1}{2}a^\dagger CTa^\dagger\right)|0\rangle\,. \tag{4.12}$$

Here we assumed

$$\lim_{n\to\infty}T^n=0\,. \tag{4.13}$$

Although we do not have a proof of this equation, it was discussed in [5] that the absolute values of the eigenvalues of T are less than one. The existence of the smooth limit to the sliver state [4] also supports the existence of this limit, since the above discussion gives an analytical explanation for the equivalence between two definitions of the sliver, namely the original definition in [6] and the oscillator representation [5].

Up to a normalization, the wedge state over the sliver state is given by

$$|n\rangle_w=\exp\left(-\frac{1}{2}s^\dagger C(-T)^{n-1}s^\dagger\right)|\Xi\rangle\,. \tag{4.14}$$

4.2 Neumann coefficients of N-string vertex

In this subsection, we will present a closed formula of the Neumann coefficients V_N^{rs} in the N-string vertex

$$|V_N\rangle=\exp\left(-\frac{1}{2}\sum_{r,s=1}^{N}a_r^\dagger V_N^{rs}a_s^\dagger\right)|0\rangle_{1\dots N}\,. \tag{4.15}$$

See [20] for the earlier discussion on the relation between the comma vertex and the Neumann coefficient.

We can easily generalize the comma representation of the 3-string vertex (2.19) to the N-string vertex:

$$|V_N\rangle = \exp\left(-\sum_{r=1}^{N} s_r^\dagger C L s_{r+1}^\dagger\right)|\Xi\rangle_{1\cdots N} = \exp\left(-\frac{1}{2}\sum_{r,s=1}^{N} s_r^\dagger \widehat{V}_N^{rs} s_s^\dagger\right)|\Xi\rangle_{1\cdots N}, \quad (4.16)$$

where $\widehat{V}_N^{rs}$ is given by

$$C\widehat{V}_N = \begin{pmatrix} 0 & L & 0 & 0 & \cdots & R \\ R & 0 & L & 0 & \cdots & 0 \\ 0 & R & 0 & L & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & R & 0 & L \\ L & 0 & \cdots & 0 & R & 0 \end{pmatrix}, \quad (4.17)$$

in other words,

$$C\widehat{V}_N^{rs} = \delta_{s-r,1} L + \delta_{r-s,1} R\,. \quad (4.18)$$

This expression of N-string vertex satisfies the required gluing relations [9]

$$\begin{aligned} {}_N\langle I|V_N\rangle_{1\cdots N} &= |V_{N-1}\rangle_{1\cdots N-1}\,, && (4.19)\\ \langle R(K,L)||V_{M+1}\rangle_{1\cdots M,K}|V_{N+1}\rangle_{L,M+1,\cdots,M+N} &= |V_{M+N}\rangle_{1\cdots M+N}\,. && (4.20)\end{aligned}$$

By going back to the original variables, we find that V_N^{rs} is given by

$$CV_N = (T + C\widehat{V}_N)(1 + C\widehat{V}_N T)^{-1}\,, \quad (4.21)$$

where $(T + C\widehat{V}_N)^{rs} = T\delta^{rs} + C\widehat{V}_N^{rs}$. For example, 4-string coefficient V_4^{rs} becomes

$$\begin{aligned} CV_4^{11} &= -CV_4^{13} = \frac{T}{1+T^2} = \frac{X}{1+X}\,,\\ CV_4^{12} &= \frac{L+T^2R}{1+T^2} = \frac{1}{2}\left(1 + \frac{Y-Z}{1+X}\right)\,,\\ CV_4^{14} &= \frac{R+T^2L}{1+T^2} = \frac{1}{2}\left(1 + \frac{Z-Y}{1+X}\right)\,. \end{aligned} \quad (4.22)$$

This expression agrees with the result in [17].

For general N, we can find the Neumann coefficients as follows. Since V_N^{rs} depends only on $s - r$ mod N, we can regard it as a vector. For the object A_s with a mod N index, we define the discrete Fourier transformation

$$A_s = \frac{1}{N}\sum_{k=1}^{N} \omega^{ks} \widehat{A}_k\,, \quad (4.23)$$

where $\omega = \exp(2\pi i/N)$. Then the convolution of two objects A_r and B_s becomes

$$\sum_{r=1}^{N} A_r B_{s-r} = \frac{1}{N}\sum_{k=1}^{N} \omega^{ks} \widehat{A}_k \widehat{B}_k \,. \tag{4.24}$$

Using the Fourier representation of the matrices in the right-hand-side of (4.21)

$$(T + C\widehat{V}_N)^{rs} = \frac{1}{N}\sum_{k=1}^{N} \omega^{k(s-r)} \Big[(T+\omega^{-k})L + (T+\omega^{k})R\Big] \,, \tag{4.25}$$

$$\Big[(1 + C\widehat{V}_N T)^{-1}\Big]^{rs} = \frac{1}{N}\sum_{k=1}^{N} \omega^{k(s-r)} \left[\frac{1}{1+T\omega^{-k}}L + \frac{1}{1+T\omega^{k}}R\right] \,, \tag{4.26}$$

the Neumann coefficients for the general N are found to be

$$\begin{aligned} CV_N^{rs} &= \frac{1}{N}\sum_{k=1}^{N} \omega^{k(s-r)} \left[\frac{T+\omega^{-k}}{1+T\omega^{-k}}L + \frac{T+\omega^{k}}{1+T\omega^{k}}R\right] \\ &= \frac{T(-T)^{[s-r]} + (-T)^{[s-r-1]}}{1-(-T)^N}L + \frac{T(-T)^{[r-s]} + (-T)^{[r-s-1]}}{1-(-T)^N}R \,, \end{aligned} \tag{4.27}$$

where

$$[s] \equiv s \bmod N \,, \qquad [s] \in \{0, 1, \ldots, N-1\} \,. \tag{4.28}$$

One can check that this form of V_N^{rs} satisfies the descent relation

$$CV_{N-1}^{rs} = CV_N^{rs} + \frac{CV_N^{rN} CV_N^{Ns}}{1 - CV_N^{11}} \,, \tag{4.29}$$

which comes from (4.19). One can also check that

$$CV_N^{rs} = CV_{2N}^{rs} + CV_{2N}^{r,s+N} = CV_{3N}^{rs} + CV_{3N}^{r,s+N} + CV_{3N}^{r,s+2N} = \cdots = \sum_{l=0}^{k-1} CV_{kN}^{r,s+lN} \,. \tag{4.30}$$

This is a generalization of the well-known relation between 3- and 6-string vertices [17].

Here we comment on the relation between the N-string vertices and the wedge states. The integral representation of Neumann coefficient is given by

$$(V_N^{rs})_{nm} = -\frac{1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{-n} w^{-m} \frac{\partial_z f_N^{(r)}(z) \partial_w f_N^{(s)}(w)}{[f_N^{(r)}(z) - f_N^{(s)}(w)]^2} \,, \tag{4.31}$$

where

$$f_N^{(r)}(z) = \tan\left(\frac{2}{N}\tan^{-1} z - \frac{\pi r}{N}\right) = S^r\left(\tan\left(\frac{2}{N}\tan^{-1} z\right)\right) \,. \tag{4.32}$$

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S is an $\mathrm{SL}(2, \mathbb{R})$ transformation

$$S(z) = \frac{z \cos \frac{\pi}{N} - \sin \frac{\pi}{N}}{z \sin \frac{\pi}{N} + \cos \frac{\pi}{N}}\,, \tag{4.33}$$

which satisfies $S^N = 1$. Since the matrix CT_N in the wedge state $|N\rangle_w$ has the same integral representation as V_N^{11}, they should be the same. This can be also understood from the defining equations $|N\rangle_w = (|0\rangle)_\star^{N-1} = {}_1\langle 0| \cdots {}_{N-1}\langle 0||V_N\rangle$. As expected, the diagonal component in (4.27) agrees with (4.8)

$$CV_N^{11} = T_N = \frac{T + (-T)^{N-1}}{1 - (-T)^N}\,. \tag{4.34}$$

5. Summary and discussions

In this paper, we analyzed the structure of the string field algebra using the so-called comma vertex. We obtained a closed formula of the N-string Neumann coefficient and showed that the diagonal element of this coefficient agrees with the width matrix of the wedge state.

It will be interesting to extend the idea of the comma vertex to the general CFT background. In fact, the 3-string vertex over the sliver state was constructed in an arbitrary CFT [12]. Up to now, its relation to the comma vertex is unclear. It is also important to study the comma representation in the ghost sector. In [5], it was argued that the 3-string vertex in the ghost sector can also be written in a comma form except for the subtlety associated with the ghost zero-modes.

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